Breakdown of shock-wave-structure solutions

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We determine, for a generic dissipative hyperbolic system of balance laws, an upper bound such that for shock velocity greater than this limit no continuous shock-wave-structure solutions may exist. These general results are applied to the old and open problem of shock waves in classical and relativistic nonequilibrium thermodynamics. In this context, for the macroscopic theories of the extended thermodynamics related to the moment Grad procedure for the Boltzmann equation we can prove that, in contrast with a recent paper [D. Jou and D. Pavon, Phys. Rev. A **44**, 6496 (1991)], this upper bound for critical Mach numbers is not influenced by adding other nonlinear terms. Moreover, taking into account the results of Weiss (Doctoral dissertation in Physics, Technical University Berlin, 1990), we can verify that our critical upper bound oscillates when the number of moments is increased. Therefore we conclude that the critical Mach number does not increase if we also consider more and more moments. As at the present the experiments do not put in evidence, also for high Mach numbers, a subshock formation in the shock structure, the natural conclusion of our result is that the shock thickness problem is not in the range of any hyperbolic continuum theory compatible with the Boltzmann equation.

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I. INTRODUCTION

The problem of giving a satisfactory explanation about the behavior of the thickness (of order of mean free path) of shock-wave structure for increasing Mach numbers, using a continuum model, is an old question. As is well known, the classical parabolic models such as Navier-Fourier or Burnett do not predict satisfactory results [1] compared with the experiments [2], while the hyperbolic models as the 13-moment Grad equations or the equivalent continuum models of extended thermodynamics [3–5] do not admit continuous solutions after a very small critical Mach number (subshock formation) [6,7]. In this last case, recently Jou and Pavon [8], taking into account that the previous hyperbolic models are linear in the nonequilibrium variables as heat flux and shear stress, have conjectured, resuming a previous opinion of Anile and Majorana [7], that for increasing critical Mach number it is necessary to add nonlinear terms in these variables. Another natural conjecture is that the critical Mach number increases and the thickness behavior becomes more satisfactory if one considers in the Boltzmann equation more and more moments instead of the usual 13 moments. This idea is sustained by similar problems as the phase velocity in the limit of high frequencies or in the light scattering in which it was necessary to consider more than 13 moments to obtain a good agreement with the experimental data [9,5].

The main goal of this paper is to prove that both the previous conjectures are not true. For this aim it is necessary to understand the mathematical general reasons of the breakdown of shock-wave-structure C^1 solutions for a generic hyperbolic dissipative system of balance laws to which the extended thermodynamics models belong. Then we are able to deduce a very simple upper bound such that for shock velocity greater than this limit no continuous shock-wave-structure solutions may exist. The first surprising result lies in the fact that this quantity is a characteristic of the associated linearized (in the neighborhood of the equilibrium unperturbed state) equations. Therefore, in the case of extended thermodynamics, in contrast with the Jou-Pavon hypothesis, this upper bound does not change by adding nonlinear nonequilibrium terms. Moreover, it coincides in the context of moments theory with the smallest characteristic eigenvalue of the linearized system that is greater than the sound velocity. Taking into account that Weiss [9] has evaluated explicitly all the characteristic eigenvalues of the linear system up to the remarkable number of 5456 moments, we are able to establish that our upper bound critical Mach number is not a monotonous function of the moments number but oscillates in the neighborhood of 1 when the moments number increases. Therefore it is impossible to get good results adding nonlinear terms, or considering more and more moments. We conclude that the behavior of the thickness as a function of Mach number is not in the range of any continuum nonequilibrium thermodynamics compatible, in the sense of moments, with the Boltzmann equation.

On the other hand, as the system of extended thermodynamics is in the form of balance laws, it is always possible to study shocks without thickness like in the nondissipative case [10]. The shock thickness remains only a microscopic phenomenon.

II. THE BALANCE LAWS SYSTEM

Let us consider a generic system of N balance laws in one space dimension:

$$\frac{\partial \mathbf{F}(\mathbf{u})}{\partial t}+\frac{\partial \mathbf{G}(\mathbf{u})}{\partial x}=\mathbf{f}(\mathbf{u}) , \tag{1}$$

where the densities $\mathbf{F}$, the fluxes $\mathbf{G}$, and the productions $\mathbf{f}$ are $\mathbb{R}^N$ column vectors depending on the space variable x and time t through the field $\mathbf{u}\equiv\mathbf{u}(x,t)\in\mathbb{R}^N$.

In the physical examples, typically in the extended thermodynamics (see Sec. V), $M<N$ equations of (1) represent conservation laws. Therefore, we suppose that the system (1) is equivalent to

$$\frac{\partial \mathbf{V}(\mathbf{u})}{\partial t}+\frac{\partial \mathbf{P}(\mathbf{u})}{\partial x}=0 , \tag{2a}$$

$$\frac{\partial \mathbf{W}(\mathbf{u})}{\partial t}+\frac{\partial \mathbf{R}(\mathbf{u})}{\partial x}=\mathbf{g}(\mathbf{u}) , \tag{2b}$$

where $\mathbf{V},\mathbf{P}\in\mathbb{R}^M$, while $\mathbf{W}$, $\mathbf{R}$, and g are vectors of $\mathbb{R}^{N-M}$:

$$\mathbf{F}\equiv\begin{bmatrix}\mathbf{V}\\ \mathbf{W}\end{bmatrix} , \quad \mathbf{G}\equiv\begin{bmatrix}\mathbf{P}\\ \mathbf{R}\end{bmatrix} , \quad \mathbf{f}\equiv\begin{bmatrix}0\\ \mathbf{g}\end{bmatrix} .$$

Moreover, we assume that it is possible to consider the field $\mathbf{u}$ constituted by a pair $\mathbf{u}\equiv(\mathbf{v},\mathbf{w})^T$ (the superscript T denotes transpose) with $\mathbf{v}\in\mathbb{R}^M$ and $\mathbf{w}\in\mathbb{R}^{N-M}$, such that

$$\mathbf{g}(\mathbf{v},0)\equiv 0 \ \ \forall\mathbf{v} , \quad \mathbf{g}(\mathbf{v},\mathbf{w})\neq 0 \ \forall\mathbf{v} , \quad \forall\mathbf{w}\neq 0 . \tag{3}$$

We call the generic state for which $\mathbf{w}=\mathbf{0}$ an equilibrium state and the $N-M$ components of $\mathbf{w}$ characterize the nonequilibrium state variables.

We require also that the system (1) is hyperbolic in the time direction and we indicate with $\lambda^{(k)}(\mathbf{u})$ $(k=1,2,\ldots,N)$ the eigenvalues of $\partial\mathbf{G}(\mathbf{u})/\partial\mathbf{u}$ with respect to the matrix $\partial\mathbf{F}(\mathbf{u})/\partial\mathbf{u}$, i.e., solutions of

$$\det\left[\frac{\partial \mathbf{G}(\mathbf{u})}{\partial \mathbf{u}}-\lambda\frac{\partial \mathbf{F}(\mathbf{u})}{\partial \mathbf{u}}\right]=0 .$$

The λ's are the characteristic velocities of the system (1) that, for the hyperbolicity assumption, are all real and finite. Moreover, we call the system

$$\frac{\partial \mathbf{V}(\mathbf{v},0)}{\partial t}+\frac{\partial \mathbf{P}(\mathbf{v},0)}{\partial x}=\mathbf{0} , \tag{4}$$

obtained by the system (2a) setting identically equal to zero the nonequilibrium variables $\mathbf{w}$, the equilibrium subsystem associated with the system (2), and we represent with $\mu^{(J)}(\mathbf{v})$ $(J=1,2,\ldots,M)$ the corresponding characteristic velocities:

$$\det\left[\frac{\partial \mathbf{P}(\mathbf{v},0)}{\partial \mathbf{v}}-\mu\frac{\partial \mathbf{V}(\mathbf{v},0)}{\partial \mathbf{v}}\right]=0 . \tag{5}$$

Of course, no relation exists, in general, between the characteristic velocities $\mu^{(J)}(\mathbf{v})$ of the equilibrium subsystem and the equilibrium values of the characteristic velocities $\lambda^{(k)}(\mathbf{v},0)$ of the full system.

III. SHOCK-WAVE STRUCTURE

As it is well known, a shock-wave structure is a regular solution of (1) depending on one variable

$$\mathbf{u}\equiv\mathbf{u}(\varphi) , \quad \varphi=x-st , \quad s=\text{const (shock velocity)} \tag{6}$$

such that

$$\lim_{\varphi\to\pm\infty}\mathbf{u}(\varphi)=\begin{cases}\mathbf{u}_0\\ \mathbf{u}_1\end{cases} , \tag{7a}$$

$$\lim_{\varphi\to\pm\infty}\frac{d\mathbf{u}}{d\varphi}=0 , \tag{7b}$$

i.e., a wave solution connecting two constant states [11].

Substituting (6) into (1), we have an ordinary differential system

$$\left[-s\frac{\partial \mathbf{F}(\mathbf{u})}{\partial \mathbf{u}}+\frac{\partial \mathbf{G}(\mathbf{u})}{\partial \mathbf{u}}\right]\frac{d\mathbf{u}}{d\varphi}=\mathbf{f}(\mathbf{u}) , \tag{8}$$

or, equivalently, from (2)

$$\frac{d}{d\varphi}[-s\mathbf{V}(\mathbf{v},\mathbf{w})+\mathbf{P}(\mathbf{v},\mathbf{w})]=\mathbf{0} , \tag{9a}$$

$$-s\frac{d}{d\varphi}\mathbf{W}(\mathbf{v},\mathbf{w})+\frac{d}{d\varphi}\mathbf{R}(\mathbf{v},\mathbf{w})=\mathbf{g}(\mathbf{v},\mathbf{w}) . \tag{9b}$$

Taking into account (7b), we obtain from (9b) evaluated at $\varphi\to\pm\infty$

$$\mathbf{g}(\mathbf{v}_0,\mathbf{w}_0)=\mathbf{g}(\mathbf{v}_1,\mathbf{w}_1)=0 , \tag{10}$$

implying, from (3), that the solutions at infinity are equilibrium solutions:

$$\mathbf{w}_0=\mathbf{w}_1=0 . \tag{11}$$

From (9a) the conservation along the process of

$$-s\mathbf{V}(\mathbf{v},\mathbf{w})+\mathbf{P}(\mathbf{v},\mathbf{w})=\mathbf{c}=\text{const} \tag{12}$$

follows and so, in particular, for $\varphi\to\pm\infty$ [see (11)]

$$\mathbf{c}=-s\mathbf{V}(\mathbf{v}_0,0)+\mathbf{P}(\mathbf{v}_0,0)=-s\mathbf{V}(\mathbf{v}_1,0)+\mathbf{P}(\mathbf{v}_1,0) . \tag{13}$$

Equation (13) coincides with the well-known Rankine-Hugoniot compatibility conditions for shocks of the equilibrium subsystem (4).

From (13), except the trivial solution $\mathbf{v}_1=\mathbf{v}_0$ (null shock) that exists always for all s, it is possible for a fixed value of $\mathbf{v}_0$ (unperturbed state) to determine the perturbed equilibrium state $\mathbf{v}_1$ as a function of the shock parameter s [12]:

$$\mathbf{v}_1\equiv\mathbf{v}_1(\mathbf{v}_0,s) . \tag{14}$$

It is known that an admissible shock (14), solution of (13), must satisfy the so-called Lax conditions [13]; i.e., in correspondence with a fixed eigenvalue μ of the equilibrium subsystem (4) one has

$$\mu_0<s<\mu_1 , \quad \lim_{s\to\mu_0}\mathbf{v}_1(\mathbf{v}_0,s)=\mathbf{v}_0 \quad [\mu_0=\mu(\mathbf{v}_0),\ \mu_1=\mu(\mathbf{v}_1)] . \tag{15}$$

In fluid dynamics the first condition represents the well-known requirement that the physical shocks are supersonic in one side and subsonic on the other one. The second condition implies that the shock passes through the null shock and so $s=\mu_0$ is a bifurcation point between the trivial solution $\mathbf{v}_1=\mathbf{v}_0$ and the solution (14). The Lax conditions guarantee, at least for weak shocks, the existence of only one family of solutions (14) for each of the M eigenvalues μ of (5) [14]. For the following it is important to point out that, for the previous considerations, the shock velocity s satisfies the inequality

$$s \ge \mu_0 \ . \tag{16}$$

For a fixed value of $\mathbf{v}_0$ and s the constant vector $\mathbf{c}$ in (12) is determined by the first equality of (13) and therefore, at least in principle, it is possible to solve (12) in $\mathbf{v}$:

$$\mathbf{v} \equiv \mathbf{v}(\mathbf{w}, \mathbf{v}_0, s) \ . \tag{17}$$

Inserting (17) in (9b) we obtain a nonlinear ordinary differential system of $N-M$ equations (depending of $M+1$ parameter s and $\mathbf{v}_0$) for the nonequilibrium vector $\mathbf{w} \in \mathbb{R}^{N-M}$ as function of φ vanishing at $\pm\infty$.

IV. BREAKDOWN OF THE SOLUTION

In this section, we prove a simple theorem permitting us to determine an upper bound, easy to evaluate, such that for shock velocity greater than this limit no continuous shock-wave-structure solutions may exist.

Theorem: We consider a system of balance laws (2) and we suppose that

$$\max_{k=1,2,\ldots,N} \lambda_0^{(k)} > \max_{J=1,2,\ldots,M} \mu_0^{(J)}$$
$$[\lambda_0 = \lambda(\mathbf{v}_0, 0), \ \mu_0 = \mu(\mathbf{v}_0)] \ , \tag{18}$$

where $\mathbf{u}_0 \equiv (\mathbf{v}_0, 0)^T$ is the unperturbed equilibrium state of a shock wave structure. For a prefixed eigenvalue $\mu_0 \in \mu_0^{(J)}$, $J = 1,2,\ldots,M$, of the equilibrium system, we start from the trivial shock for which $s = \mu_0$ and we increase s satisfying the Lax condition $s > \mu_0$. Then there exists always a finite critical value s_c of the shock velocity such that a breakdown of the shock-structure solution happens. In particular, it is impossible to have a C^1 solution for

$$s \ge \tilde{\lambda}_0 \ , \tag{19}$$

where $\tilde{\lambda}_0$ is the smallest $\lambda_0^{(k)}$ greater than μ_0.

Proof: For a fixed value of the parameter s, at least in a right neighborhood of μ_0 (weak shocks), it is possible to apply also in this case for the hyperbolic systems the result of Kopell and Howard [15] (motivated by the parabolic problem) guaranteeing for weak shocks the existence of a C^1 solution of (8) satisfying the boundary conditions (7). Let us now increase s and suppose that, for a given $s_* > \mu_0$, there exists still a C^1 solution $\mathbf{u}_*(\varphi)$ of (8) connecting the two equilibrium states $\mathbf{u}_0 \equiv (\mathbf{v}_0, 0)^T$ and $\mathbf{u}_1 \equiv (\mathbf{v}_1(\mathbf{v}_0, s), 0)^T$. Considering (8) and the fact that the derivative of this solution remains bounded, the matrix

$$\left[\frac{\partial \mathbf{G}(\mathbf{u})}{\partial \mathbf{u}} - s_* \frac{\partial \mathbf{F}(\mathbf{u})}{\partial \mathbf{u}} \right]_{\mathbf{u}=\mathbf{u}_*}$$

must be not singular at least for all $\varphi \in]-\infty, +\infty[$ for which the production $\mathbf{f}$ does not vanish. Therefore in these points all the eigenvalues $\lambda_* = \lambda[\mathbf{u}_*(\varphi)]$ evaluated in $\mathbf{u}_*$ are different from s_*.

The impossibility that a characteristic velocity coincides with the shock velocity is surely verified in the limit $\varphi \to +\infty$. In fact, since at $+\infty$ the production $\mathbf{f}$ vanishes, the system (8) becomes a linear algebraic homogeneous system for the derivative of $\mathbf{u}$. So recalling (7b), as from (8) it is necessary also in this case that the matrix (evaluated in $\mathbf{u}_0$) preceding the derivative is not singular, we have $s_* \ne \lambda_0^{(k)}$ $\forall$ $k = 1,2,\ldots,N$, i.e., [16]

$$s_* < \lambda_0^{(1)} \ , \tag{20a}$$

$$\lambda_0^{(k)} < s_* < \lambda_0^{(k+1)} \quad (1 \le k \le N) \ , \tag{20b}$$

or

$$s_* > \lambda_0^{(N)} \ . \tag{20c}$$

On the other hand, we are interested in the C^1 solution obtained by increasing with continuity $s (s > \mu_0)$ starting from the trivial case $s = \mu_0$ and therefore only one of the previous inequalities (20a) or (20b) must be true, while (20c) is impossible taking into account the assumption (18). We conclude that, if there exists a C^1 solution of (8) with boundary conditions (7) satisfying the Lax conditions (15), then

$$\mu_0 \le s < \tilde{\lambda}_0 \ , \tag{21}$$

where $\tilde{\lambda}_0$ is the smallest of λ_0 appearing in (20a) and (20b) that is greater than μ_0. Therefore it is impossible to have a C^1 solution when

$$s \ge \tilde{\lambda}_0 \ . \tag{22}$$

Important Remarks

Remark 1: As is well known the characteristic eigenvalues evaluated at equilibrium λ_0, and in particular $\tilde{\lambda}_0$, coincide with the characteristic velocities of the linear system obtained linearizing the system (1) in the neighborhood of the equilibrium state $\mathbf{u}_0$. They also are coincident with the phase velocities of the linearized problem in the limit of high frequency (see, e.g., [17]). Therefore, since our problem is fully nonlinear, the critical shock velocity s_c $(\mu_0 < s_c \le \tilde{\lambda}_0)$ may depend on the nonequilibrium fields, but, according to our result, its maximum value $\tilde{\lambda}_0$ results, on the contrary, independent, i.e., $\tilde{\lambda}_0$ does not change by modifying or adding nonlinear nonequilibrium terms in $\mathbf{w}$.

Remark 2: We observe that the only exceptional case in which an upper critical limit is unknown occurs when the condition (18) is violated (this circumstance has not been verified, to our knowledge, in the physical applications until now) and we choose as μ_0 the greatest of the μ's.

Remark 3: We note finally that $\tilde{\lambda}_0$ is of immediate evaluation. We need only the eigenvalues $\mu_0^{(J)}$ of the equilibrium subsystem and the characteristic eigenvalues of the system (1) $\lambda_0^{(k)}$ evaluated in the equilibrium state and to identify the smallest of $\lambda_0^{(k)}$ greater than μ_0. This is very simple with respect to finding numerically the precise critical value s_c for which the breakdown takes place. In the next section we apply these results to the case of extended thermodynamics.

V. THE EXTENDED THERMODYNAMICS CASE

We have in the three-dimensional case thirteen balance laws reducing, in one space dimension, to five of the form (1): three equations, of the form (2a), are the conservation equations of mass, momentum, and energy and the remaining two, which are a particular case of (2b), describe the evolution equations of the heat flux q and the shear stress σ [3,5]:

$$\frac{\partial}{\partial t}\rho+\frac{\partial}{\partial x}(\rho v)=0 , \tag{23}$$

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho v^2+p-\sigma)=0 , \tag{24}$$

$$\frac{\partial}{\partial t}(\rho v^2+3p)+\frac{\partial}{\partial x}(\rho v^3+5pv-2\sigma v+2q)=0 , \tag{25}$$

$$\frac{\partial}{\partial t}\left[\frac{2}{3}\rho v^2-\sigma\right]+\frac{\partial}{\partial x}\left[\frac{2}{3}\rho v^3+\frac{4}{3}pv-\frac{7}{3}\sigma v+\frac{8}{15}q\right]=\tau_0\sigma , \tag{26}$$

$$\frac{\partial}{\partial t}(2q+5pv-2\sigma v+\rho v^3)+\frac{\partial}{\partial x}\left[\rho v^4+5\frac{p^2}{\rho}-7\frac{\sigma p}{\rho}+\frac{32}{5}qv+v^2(8p-5\sigma)\right]=2\tau_0 v\sigma-\tau_1 q . \tag{27}$$

We choose as field $\mathbf{u}\in\mathbb{R}^5$ the mass density ρ, the velocity in the x direction v, the absolute temperature Θ, the one-dimensional components of heat flux q, and the shear stress σ. Moreover, for a monatomic gas the pressure p and the internal energy e satisfy the relation $p=\frac{2}{3}\rho e=k\rho\Theta$, while τ_0 and τ_1 are positive parameters related to the heat conductivity and the viscosity coefficient [3,5]. As is well known, the previous system coincides for monatomic gases with the one obtained by Grad from the Boltzmann equation using 13 moments.

This example belongs to the previous general framework now having

$$N=5 , \quad M=3 , \quad \mathbf{v}\equiv(\rho,v,\Theta)^T , \quad \mathbf{w}\equiv(q,\sigma)^T . \tag{28}$$

The two equilibrium states $\mathbf{u}_0$ (unperturbed) and $\mathbf{u}_1$ (perturbed) are in, according to the condition (11),

$$\begin{aligned}&\mathbf{u}_0\equiv(\rho_0,v_0=0,\ \Theta_0,q_0=0,\ \sigma_0=0)^T ,\\&\mathbf{u}_1\equiv(\rho_1,v_1,\Theta_1,q_1=0,\sigma_1=0)^T .\end{aligned} \tag{29}$$

The values ρ_1, Θ_1, and v_1 are related through the well-known (see, e.g., [18]) Rankine-Hugoniot equations (13) to ρ_0, Θ_0, and depend on the shock parameter $M_0=s/c_0$ [with $c_0=\sqrt{(5/3)k\Theta_0}$ denoting the sound velocity].

In this case the equilibrium subsystem (4) coincides with Eqs. (23)–(25) when $q\equiv0$ and $\sigma\equiv0$, i.e., with the Euler equations, and so

$$\mu_0^{(1)}=-c_0 , \quad \mu_0^{(2)}=0 , \quad \mu_0^{(3)}=c_0 . \tag{30}$$

Instead, the equilibrium characteristic eigenvalues of the full system are (see [19,5])

$$\begin{aligned}&\lambda_0^{(1)}=-1.65c_0 , \quad \lambda_0^{(2)}=-0.62c_0 , \quad \lambda_0^{(3)}=0 ,\\&\lambda_0^{(4)}=0.62c_0 , \quad \lambda_0^{(5)}=1.65c_0 .\end{aligned} \tag{31}$$

Choosing the shock traveling in the $x>0$ direction, we have

$$\mu_0=c_0 , \quad \tilde{\lambda}_0=\lambda_0^{(5)}=1.65c_0 , \tag{32}$$

and therefore C^1 solutions can exist only for Mach numbers such that

$$1\le M_0<1.65 , \tag{33}$$

while for

$$M_0\ge1.65 , \tag{34}$$

C^1 solutions are forbidden. So we find again, in an immediate way, the well-known Grad result [6,7]. It is interesting to note that the critical Mach number determined by Grad [6] and Anile and Majorana [7] through a heavy numerical integration of the system (8) coincides in this case with our upper limit (34).

As consequence of remark 1 of the preceding section, if the critical Mach number depends on the nonequilibrium variables, it is impossible to change its upper limit (34) by adding nonlinear terms in q or σ, in contrast, therefore, with one of the conjectures of [8]. In our opinion, in Ref. [8], where a continuum model is considered in the spirit of the extended irreversible thermodynamics, the apparent increase of the critical Mach number is due to the fact that the authors in reality modify the system (23)–(27) also in the linear part of the differential operator with the consequence that they change the λ_0's too.

In the relativistic case we do not know a precise evaluation of s_c but only some general considerations can be found in the papers [20–22]. Besides, in [22] the authors deduce for the old Müller-Israel-Stewart [23] relativistic theory a maximum upstream Mach number corresponding to the characteristic equilibrium velocities of the fluid. This last result, obtained in this particular case also if the system considered by these authors is not in the form of a balance law system (1), is in perfect agreement with our general result whose validity regards a generic hyperbolic system of balance laws. In the case of the modern theory of relativistic fluids developed by Liu, Müller, and Ruggeri [4], considering a system of 14 balance laws and having a precise correspondence with the 14 moments theory arising from the relativistic Boltzmann equation, Seccia and Strumia [24] have evaluated the characteristic velocities λ_0's and therefore it is a simple matter to deduce also in this complicated case the $\tilde{\lambda}_0$ such that for s greater than this quantity it is impossible to have a continuous and differentiable shock-wave-structure solution. Since the expression of $\tilde{\lambda}_0$ is a heavy formula [24], we consider here only the ultrarelativistic limit case where

$$\mu_0=\sqrt{\tfrac{1}{3}}c$$

and the (positive) λ_0 are

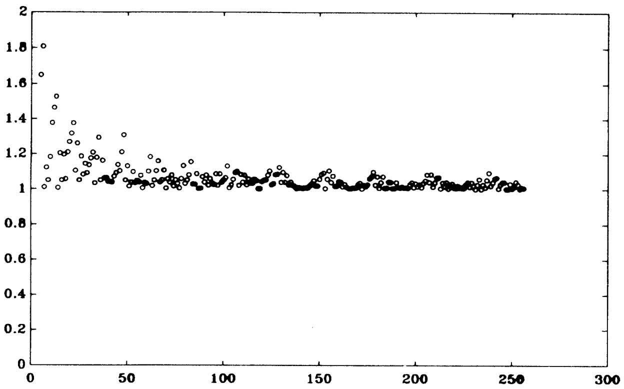

FIG. 1. Dimensionless critical upper bound $\tilde{\lambda}_0/c_0$ (vertical axis) vs the moment number (horizontal axis).

$$\lambda_0=\sqrt{\tfrac{1}{5}}c\ ,\quad \sqrt{\tfrac{1}{3}}c\ ,\quad \sqrt{\tfrac{3}{5}}c\ ,$$

where c is the light velocity. In this circumstance, C^1 solutions may exist in the interval

$$\sqrt{\tfrac{1}{3}}c<s<\sqrt{\tfrac{3}{5}}c\ ,$$

and no C^1 solution exists for $s\geq\sqrt{\tfrac{3}{5}}c$. This last result coincides with the value found in [22] using the old model of relativistic extended irreversible thermodynamics.

VI. MORE AND MORE MOMENTS

As we have pointed out in the Introduction, a natural question is to ask what happens by considering, in the classical context, more than 13 moments. In this case the conservation laws always remain the usual ones and therefore μ_0 is still equal to c_0, the λ_0's changing every time we increase the moments number. Fortunately, Weiss [9], studying the problem of the behavior of phase velocity in the limit of high frequency, has evaluated all the λ_0's until 5456 moments: therefore it is a simple matter to establish, for a fixed number of moments, the minimum of eigenvalues greater than $\mu_0=c_0$. In Fig. 1 we plot our upper bound $\tilde{\lambda}_0/c_0$ for increasing moments (take into account that 5456 moments correspond, in the one-dimensional case, to 256 equations). From Fig. 1 it is evident that $\tilde{\lambda}_0/c_0$ oscillates with a small damping over 1. Therefore, if the moments number increases, we always have a breakdown of the solution for critical Mach numbers a little more greater than 1.

Then, we conclude that in a continuum approach, independently by the number of variables, it is impossible to find continuous solutions and only shocks without thickness (weak solutions) have validity according with previous similar opinions (see, e.g., [25]) and the assumptions that in the continuum limit the mean free path tends to zero.

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